Bounded Derivations on Uniform Roe Algebras

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Abstract

In this paper we prove that if $C^*_u(X)$ is a nuclear uniform Roe algebra associated to a bounded geometry metric space X, then all bounded derivations on $C^*_u(X)$ are inner.

1 Introduction

Let A be a C^*-algebra. A *derivation* of A is a linear map $\delta : A \to A$ satisfying $\delta(ab) = a\delta(b) + \delta(a)b$. In this paper, we always assume that our derivations are defined on all of A, and are thus bounded by a fundamental result of Sakai [9]. A derivation δ of A is *inner* if there exists d in the multiplier algebra $M(A)$ of A such that $\delta(a) = ad - da$ for all $a \in A$. Let us say that a C^*-algebra A *only has inner derivations* if all (bounded) derivations are inner.

Motivated by the needs of mathematical physics and the study of one-parameter automorphism groups, it is interesting to study whether all derivations are inner for a particular C^*-algebra. In the 1970s, a complete solution to this problem was obtained in the separable case via the work of several authors. The definitive result was obtained by Akemann and Pedersen [1]: they showed that a separable C^*-algebra has only inner derivations if and only if it isomorphic to a C^*-algebra of the form

$$C \oplus \bigoplus_{i \in I} S_i, \tag{1}$$

where C is continuous trace (possibly zero), and each S_i is simple (possibly zero). Thus in particular all separable commutative, and all separable simple, C^*-algebras only have inner derivations. However, one might reasonably say that most separable C^*-algebras admit non-inner derivations.

For non-separable C^*-algebras the picture is murkier. It is well-known that there are non-separable C^*-algebras, not of the form in line (1), that only have inner derivations: perhaps most famously, Sakai [10] has shown this for all von Neumann algebras. See also for example [5, page 123] for some examples that are not von Neumann algebras, nor of the form in line (1), and that only have inner derivations.

Our goal in this paper is to give a new class of examples that only have inner derivations: nuclear uniform Roe algebras. Uniform Roe algebras are a well-studied class of non-separable C^*-algebras associated to metric spaces: see Section 2 below for basic definitions. They were originally introduced for index-theoretic purposes, but are now studied for their own sake as a bridge between C^*-algebra theory and coarse geometry, as well as having interesting applications to single operator theory and mathematical physics amongst other things. Due to the presence of $\ell^\infty(X)$ as a diagonal MASA[1] they have a somewhat von Neumann algebraic flavor, but are von Neumann algebras only in the trivial finite-dimensional case. They are also essentially never of the form in line (1). Moreover, in many ways they are quite tractable as C^*-algebras, often having good regularity properties such as nuclearity.

Here is our main theorem.

Theorem 1.1. *Nuclear uniform Roe algebras associated to bounded geometry metric spaces only have inner derivations.*

Key ingredients in the proof come from recent groundbreaking work on uniform Roe algebras of Braga-Farah [2], Špakula-Tikuisis [13], and Špakula-Zhang [14].

We leave the following natural questions open.

- Do all uniform Roe algebras only have inner derivations?
- The fact that all derivations on A are inner can be restated as saying that the first Hochschild cohomology group $H^1(A, A)$ vanishes. For A a (nuclear) uniform Roe algebra, do all the higher groups $H^n(A, A)$ vanish? See [11] for a survey of this problem in the case that A is a von Neumann algebra.

[1] In the sense of Kumjian: see [7].

2 Definitions and background results

In this section, we recall the definition of uniform Roe algebras. We also recall two classical results: the geometric characterization of when uniform Roe algebras are nuclear, due to Skandalis, Tu, and Yu; and a classical result of Kadison saying that all derivations are spatially implemented.

Inner products are linear in the first variable. For a Hilbert space $\mathcal{H}$ we denote the space of bounded operators on $\mathcal{H}$ by $\mathscr{B}(\mathcal{H})$, and the space of compact operators by $\mathscr{K}(\mathcal{H})$. The commutator of $a, b \in \mathscr{B}(\mathcal{H})$ is denoted by $[a, b] := ab - ba$.

The Hilbert space of square-summable sequences on a set X is denoted $\ell^2(X)$, and the canonical basis of $\ell^2(X)$ will be denoted $(\vartheta_x)_{x\in X}$ (we reserve δ for derivations). For $a \in \mathscr{B}(\ell^2(X))$ we define its matrix entries by

$$a_{xy} := \langle a\vartheta_x, \vartheta_y\rangle .$$

Definition 2.1 (propagation, uniform Roe algebra)**.** Let X be a metric space and $r \geq 0$. An operator $a \in \mathscr{B}(\ell^2(X))$ *has propagation at most* r if $a_{xy} = 0$ whenever $d(x, y) > r$ for all $(x, y) \in X \times X$. In this case, we write $\mathrm{prop}(a) \leq r$. The set of all operators with propagation at most r is denoted $\mathbb{C}_u^r[X]$. We define

$$\mathbb{C}_u[X] := \{a \in \mathscr{B}(\ell^2(X)) : \mathrm{prop}(a) < \infty\};$$

it is not difficult to see that this is a $*$-algebra. The *uniform Roe algebra*, denoted $C_u^*(X)$, is defined to be the norm closure of $\mathbb{C}_u[X]$.

Definition 2.2 (*ϵ-R-approximated*)**.** Let X be a metric space. Given $\epsilon > 0$ and $r > 0$, an operator $a \in \mathscr{B}(\ell^2(X))$ can be *ϵ-r-approximated* if there exists an $b \in \mathbb{C}_u^r[X]$ such that $\|a - b\| < \epsilon$.

We will exclusively be interested in uniform Roe algebras associated to bounded geometry metric spaces as in the next definition.

Definition 2.3 (bounded geometry)**.** A metric space X is said to have *bounded geometry* if for every $r \geq 0$ there exists an $N_r \in \mathbb{N}$ such that for all $x \in X$, the ball of radius r about x has at most N_r elements.

We will need the following theorem, which translates nuclearity of $C_u^*(X)$ into a more geometrically useful form. The result is due to Skandalis, Tu, and Yu: see [12, Theorem 5.3]. See also [3, Theorem 5.5.7] for a proof that does not use groupoid theory. We will not need the definition of property A here, just some consequences of it: see [17, Definition 2.1] for the original definition.

Theorem 2.4. *Let X be a bounded geometry metric space. Then $C_u^*(X)$ is nuclear if and only if X has property A.* $\square$

Finally in this section, we recall a general fact about derivations.

Definition 2.5 (spatial derivation)**.** Let $A \subseteq \mathscr{B}(\mathcal{H})$ be a concrete C^*-algebra. A derivation δ of A is *spatial* if there is a bounded operator $d \in \mathscr{B}(\mathcal{H})$ such that $\delta(a) = [a, d]$.

The following is due to Kadison [6, Theorem 4].

Theorem 2.6. *Let $A \subseteq \mathscr{B}(\mathcal{H})$ be a concrete C^*-algebra. Then every derivation on A is spatial.* $\square$

Note that the uniform Roe algebra $C_u^*(X)$ always contains the compact operators on $\ell^2(X)$. For a concrete C^*-algebra $A \subseteq \mathscr{B}(\mathcal{H})$ containing the compact operators $\mathscr{K}(\mathcal{H})$, there are simpler proofs of Theorem 2.6 available: see for example [4, Corollary 3.4 and Remark on page 284].

3 Proof of main result

In this section, we start by summarizing facts we need from recent work of Špakula-Tikuisis, Špakula-Zhang, and Braga-Farah. We then prove Theorem 1.1.

The first main theorem we need is due to Špakula-Zhang [14], building on ideas of Špakula-Tikuisis [13].

The precise statement below is a straightforward consequence of the equivalence of (i) and (iv) in [14, Theorem 3.3]. For the statement, if f is an element of $\ell^\infty(X)$, we consider f as a bounded operator on $\ell^2(X)$ by multiplication, and for $L > 0$ we write $\mathrm{Lip}(f) \leq L$ if for every $x, y \in X$ we have $|f(x) - f(y)| \leq L d(x, y)$, i.e. if the Lipschitz constant is at most L. Also, if B is a C^*-algebra, we write B_1 for the closed unit ball $\{b \in B \mid \|b\| \leq 1\}$.

Theorem 3.1. *Let X be a bounded geometry metric space with property A. Then an operator $a \in \mathscr{B}(\ell^2(X))$ is in $C_u^*(X)$ if and only if for any $\epsilon > 0$ there exists $L > 0$ such that whenever $f \in \ell^\infty(X)_1$ satisfies Lip$(f) \leq L$, we have $\|[f, a]\| \leq \epsilon$.* $\square$

The next result is due to Braga and Farah [2, Lemma 4.9]. To state it, let $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$ denote the closed unit disk, and for a set I, let $\mathbb{D}^I$ denote the usual product space of functions $I \to \mathbb{D}$. We write elements of $\mathbb{D}^I$ as tuples $\overline{\lambda} = (\lambda_i)_{i \in I}$.

Lemma 3.2. *Let (X, d) be a metric space with bounded geometry, and let I be a countable set. Suppose that $(a_i)_{i \in I}$ is a family of finite rank operators in $C^*_u(X)$ such that for every $\overline{\lambda} \in \mathbb{D}^I$ the series $\sum_{i \in I} \lambda_i a_i$ converges strongly to an operator $a_{\overline{\lambda}} \in C^*_u(X)$. Then for every $\epsilon > 0$ there exists $r > 0$ such that $a_{\overline{\lambda}}$ can be ϵ-r-approximated for all $\overline{\lambda} \in \mathbb{D}^I$.* □

The following basic lemma is essentially folklore: see for example [16, Lemma 8.1] for a proof.

Lemma 3.3. *Let X be a metric space with bounded geometry, and for each $r > 0$ let N_r be as in Definition 2.3. Then for $a \in \mathbb{C}^r_u[X]$ we have*

$$\|a\| \leq N_r \sup_{x,y \in X} |a_{xy}| . \qquad \square$$

The last lemma we need is probably well-known to experts. We are not aware of the precise statement appearing anywhere in the literature, so provide a proof.

Lemma 3.4. *Let X be a metric space with bounded geometry having property A, and let $m, r, \epsilon > 0$. Then there exists $s > 0$ with the following property. For all $a \in C^*_u(X)$ with norm at most m and that is ϵ-r approximated there exists $b \in \mathbb{C}^s_u[X]$ such that $\|a - b\| \leq 3\epsilon$, and such that $|b_{xy}| \leq |a_{xy}|$ for all $x, y \in X$.*

Proof. Fix $r, \epsilon > 0$, and let N_r be as in Definition 2.3. Since X has property A, [15, Theorem 1.2.4, (8)] (see also [15, Definitions 3.2.1] for terminology) gives $s > 0$ and a positive type kernel $k : X \times X \to \mathbb{R}$ such that $k(x, x) = 1$ for all $x \in X$, such that $k(x, y) = 0$ whenever $d(x, y) > s$, and such that $|1 - k(x, y)| \leq \epsilon/((m + \epsilon)N_r)$ whenever $d(x, y) \leq r$. Then using either [8, Lemma 11.17] or [3, Theorem D.3], there exists a unital completely positive map

$$M_k : \mathscr{B}(\ell^2(X)) \to \mathscr{B}(\ell^2(X))$$

satisfying

$$(M_k a)_{xy} = k(x, y)a_{xy}$$

for all $a \in \mathscr{B}(\ell^2(X))$. Hence in particular, M_k takes image in $\mathbb{C}_u^s[X]$, and moreover if $c \in \mathbb{C}_u^r[X]$, then Lemma 3.3 gives that

$$\begin{aligned}\|M_k(c) - c\| &\leq N_r \sup_{d(x,y)\leq r} |M_k(c)_{xy} - c_{xy}| = N_r \sup_{d(x,y)\leq r} |1 - k(x,y)||c_{xy}| \\ &\leq \|c\| \frac{\epsilon}{m+\epsilon}.\end{aligned}$$

Let now $a \in C_u^*(X)$ be ϵ-r approximated, so there exists $c \in \mathbb{C}_u^r[X]$ such that $\|a - c\| \leq \epsilon$ and so in particular $\|c\| \leq m + \epsilon$. Set $b := M_k a$, so b is in $\mathbb{C}_u^s[X]$. Putting the discussion so far together, we have

$$\|a - b\| \leq \|a - c\| + \|c - M_k(c)\| + \|M_k(c - a)\| \leq 3\epsilon.$$

Finally, note that for each $x, y \in X$, $|k(x,y)| \leq k(x,x)^{1/2} k(y,y)^{1/2}$ by the fact that k is positive type, and the (proof of the) Cauchy-Schwarz inequality. Hence for each $x, y \in X$, $|k(x,y)| \leq 1$, and so $|b_{xy}| = |k(x,y)a_{xy}| \leq |a_{xy}|$, completing the proof. □

We are now ready for the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\delta : C_u^*(X) \to C_u^*(X)$ be a derivation. Theorem 2.6 implies that δ is spatially implemented, so there is $d \in \mathscr{B}(\ell^2(X))$ such that $\delta(a) = [a, d]$ for all $a \in C_u^*(X)$. To prove Theorem 1.1, we will show that d is actually in $C_u^*(X)$ using Theorem 3.1.

For each $x \in X$, let p_x be the rank one projection onto the span of the canonical basis element ϑ_x. For $f \in \ell^\infty(X)_1$ (considered as a multiplication operator on $\ell^2(X)$), write f as a sum

$$f = \sum_{x \in X} f(x) p_x$$

(convergence in the strong operator topology). Then using strong continuity of subtraction, and separate strong continuity of multiplication on bounded sets,

$$[f, d] = \Big[\sum_{x \in X} f(x) p_x, d\Big] = \sum_{x \in X} f(x)[p_x, d].$$

On the other hand, by assumption that δ is a derivation on $C_u^*(X)$, $[f, d]$ is in $C_u^*(X)$ for all $f \in \ell^\infty(X)$. It follows that if we set $I = X$ and for each $x \in X$ we set $a_x := [p_x, d]$, then the collection $(a_x)_{x \in X}$ satisfies the assumptions of Lemma

3.2. Hence for every $\epsilon > 0$ there exists $r > 0$ such that for every $f \in \ell^\infty(X)_1$, the operator $[f, d]$ can be $(\epsilon/4)$-r approximated.

Hence by Lemma 3.4 (with $m = 2\|d\|$, which is an upper bound for $\|[f, d]\|$ for all $f \in \ell^\infty(X)_1$) there exists $s > 0$ such that for all $f \in \ell^\infty(X)_1$ there exists $d^f \in \mathbb{C}_u^s[X]$ such that $\left\|d^f - [f, d]\right\| \leq 3\epsilon/4$ and such that $|d^f_{xy}| \leq \left|[f, d]_{xy}\right|$ for all $x, y \in X$. Note that for any $f \in \ell^\infty(X)_1$,

$$\begin{aligned}\sup_{x,y\in X} |d^f_{xy}| = \sup_{d(x,y)\leq s} |d^f_{xy}| \leq \sup_{d(x,y)\leq s} |[f, d]_{xy}| = \sup_{d(x,y)\leq s} |f(x) - f(y)||d_{xy}| \\ \leq \mathrm{Lip}(f)s\|d\|.\end{aligned}$$

It follows from Lemma 3.3 that

$$\|[d, f]\| \leq \|d^f - [d, f]\| + \|d^f\| \leq 3\epsilon/4 + N_s\mathrm{Lip}(f)s\|d\|.$$

for all $f \in \ell^\infty(X)_1$. Hence if $L < \epsilon(4N_s\|d\|s)^{-1}$, then whenever $\mathrm{Lip}(f) \leq L$, we get $\|[d, f]\| < \epsilon$. Theorem 3.1 now completes the proof. $\square$

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